Towards a more realistic description of interferometry

Divya Oberoi

div@ncra.tifr.res.in

19 Nov 2024

Form of the observed electric field

- $\vec{E}(\vec{R}, t)$
- $\vec{E}_\nu(\vec{R}, t) = \vec{E}_\nu(\vec{R})e^{-i\omega t}$
- $E_\nu(\vec{r}) = \int\int\int P_\nu(\vec{R}, \vec{r})\vec{E}_\nu(\vec{R})dx\ dy\ dz$
  where $P_\nu(\vec{R}, \vec{r})$ is the *propagator* from $\vec{R}$ to $\vec{r}$.

- Assumption 1 - No polarization (scalar field)
- Assumption 2 - Sources lie on a *Celestial sphere*
  - 3D $\rightarrow$ 2D (though embedded in 3D space)
- Assumption 3 - There is no additional emission, absorption, scattering inside the Celestial sphere.
  - So we only have to describe the distribution of sources of electric field at this surface.

- $E_\nu(\vec{r}) = \int \mathcal{E}_\nu(\vec{R}) \, \frac{e^{\frac{2\pi i \nu |\vec{R}-\vec{r}|}{c}}}{|\vec{R}-\vec{r}|} \, \mathrm{dS}$

  where dS - surface area element on the celestial sphere.

Spatial Coherence

- $V_\nu(\vec{r_1}, \vec{r_2}) = \langle E_\nu(\vec{r_1})\ E^*_\nu(\vec{r_2})\rangle$
- $V_\nu(\vec{r_1}, \vec{r_2}) = \left\langle \int\int \mathcal{E}_\nu(\vec{R_1})\ \mathcal{E}^*_\nu(\vec{R_2})\ \frac{e^{\frac{2\pi i\nu|\vec{R_1}-\vec{r_1}|}{c}}}{|\vec{R_1}-\vec{r_1}|}\ \frac{e^{\frac{-2\pi i\nu|\vec{R_2}-\vec{r_2}|}{c}}}{|\vec{R_2}-\vec{r_2}|}\ dS_1\ dS_2 \right\rangle$
- Assumption 4 - Emission is spatially incoherent
  - $\left\langle \mathcal{E}_\nu(\vec{R_1})\ \mathcal{E}^*_\nu(\vec{R_2})\right\rangle = 0$ for $R_1 \neq R_2$

Spatial Coherence Function

- $V_\nu(\vec{r_1}, \vec{r_2}) = \int \left\langle |\mathcal{E}_\nu(\vec{R})|^2 \right\rangle \; |\vec{R}|^2 \; \frac{e^{\frac{2\pi i\nu|\vec{R}-\vec{r_1}|}{c}}}{|\vec{R}-\vec{r_1}|} \; \frac{e^{\frac{-2\pi i\nu|\vec{R}-\vec{r_2}|}{c}}}{|\vec{R}-\vec{r_2}|} \; dS$
- $V_\nu(\vec{r_1}, \vec{r_2}) = \int \mathcal{B}_\nu(\vec{s}) \; e^{\frac{-2\pi i\nu\vec{s}.(\vec{r_1}-\vec{r_2})}{c}} \; d\Omega$

  where $\vec{s} = \frac{\vec{R}}{|\vec{R}|}$; $\quad \mathcal{B}_\nu(\vec{s}) = \left\langle |\mathcal{E}_\nu(\vec{R})|^2 \right\rangle \; |\vec{R}|^2$;

  $d\Omega = |\vec{R}|^2 \; dS$ and ignoring terms of order $|r/R|$

- Also known as *Spatial Autocorrelation Function*

Fourier inversion for synthesis imaging

- $V_\nu(u, v, w) = \int \int \mathcal{B}_\nu(l, m) \frac{e^{-2\pi i(ul+vm+wn)}}{\sqrt{1-l^2-m^2}} \, dl \, dm$
- Components of $\vec{s}$ are $(l, m, \sqrt{1 - l^2 - m^2})$.
- To get to a proper FT relationship - get rid of *wn* term in the exponential (Assumption 5)
  - Let's confine all our measurements to preferred plane such that $\vec{r_1} - \vec{r_2} = \lambda(u, v, w = 0)$.
  - Small field-of-view - $(\sqrt{1 - l^2 - m^2})w \approx -\frac{1}{2}(l^2 + m^2)w << (ul + vm)$

Inverse Fourier transform for synthesis imaging

- $I_\nu^D(l,m) = \int\int V_\nu(u,v)\ S(u,v)\ e^{2\pi i(ul+vm)} du\ dv$
  $S(u,v) =$ Sampling function
  $I_\nu^D(l,m) =$ Dirty image
  $I_\nu^D(l,m) = I_\nu \star PSF$
- $PSF(l,m) = \int\int S(u,v)\ e^{2\pi i(ul+vm)} du\ dv$

Effect of the Antenna reception pattern/ image plane effects

- $V_\nu(u, v) = \int\int \mathcal{B}_\nu(l, m)\, \mathcal{A}_\nu(l, m)\, \frac{e^{-2\pi i(ul+vm)}}{\sqrt{1-l^2-m^2}}\, dl\, dm$
  $\mathcal{B}_\nu(l, m) =$ Sky brightness distribution
  $\mathcal{A}_\nu(l, m) =$ Antenna reception pattern
- Other image plane effects - all atmospheric propagation effects
  $V_\nu(u, v) =$
  $\int\int \mathcal{B}_\nu(l, m)\, \mathcal{I}_\nu(l, m)\, \mathcal{A}_\nu(l, m)\, \frac{e^{-2\pi i(ul+vm)}}{\sqrt{1-l^2-m^2}}\, dl\, dm$
  $\mathcal{I}_\nu(l, m) =$ Ionospheric phase screen

Response of an interferometer

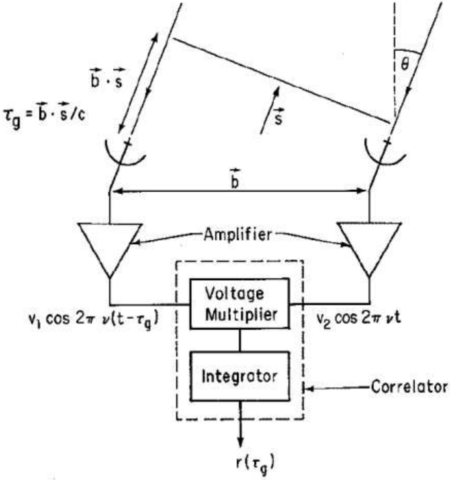


Figure 2-1. Simplified schematic diagram of a two-element interferometer.

Response of an interferometer

- Geometric delay - $\tau_g = \frac{\vec{b}.\vec{s}}{c}$
- Correlator output - $r(\tau_g) =< V_1(t)\ V_2(t) >$
- $V_1(t) = v_1\ cos\ 2\pi\nu(t - \tau_g);\ V_2(t) = v_2\ cos\ 2\pi\nu t;$
- $r(\tau_g) = v_1 v_2\ cos\ 2\pi\nu\tau_g$

Response in terms of the Brightness distribution

- $dr = A(\vec{s})\ B(\vec{s})\ \Delta\nu\ \Delta\Omega\ cos\ 2\pi\nu\tau_g$
  $A(\vec{s})$ = Effective collecting area in the direction of the source
  $B(\vec{s})$ = Brightness distribution ($W\ m^{-2}\ Hz^{-1}\ sr^{-1}$)
- $r(\tau_g) = \int_\Omega A(\vec{s})\ B(\vec{s})\ \Delta\nu\ cos\ 2\pi\nu\tau_g\ d\Omega$
- $r(\tau_g) = \Delta\nu\ \int_\Omega A(\vec{s})\ B(\vec{s})\ cos\ \frac{2\pi\nu\vec{b}\cdot\vec{s}}{c}\ d\Omega$

Phase Tracking Center

- $\vec{s} = \vec{s_0} + \vec{\sigma}$

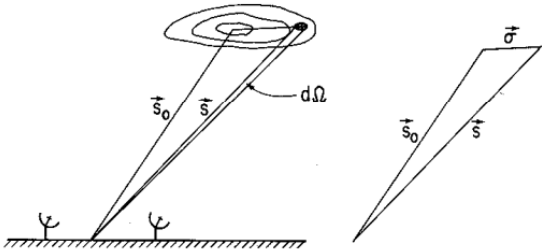


Figure 2-2. Position vectors used in deriving the interferometer response to a source. The source is represented by the contours of radio brightness $I(\mathbf{s})$ on the sky.

- $r(\sigma) = \Delta\nu \int_{\Omega} A(\vec{s})\ B(\vec{s})\ cos\ \frac{2\pi\nu\vec{b}\cdot(\vec{s_0}+\vec{\sigma})}{c}\ d\Omega$
- …

Visibility

- $V = |V|\ e^{i\phi_V} = \int_{\Omega} A_N(\vec{\sigma})\ B(\vec{\sigma})\ e^{-2\pi i\ \nu\ \vec{b}.\vec{\sigma}/c}\ d\Omega$
- $A_N(\vec{\sigma}) = A(\vec{\sigma})/A_0$
- ...
- $r = A_0\ \Delta\nu\ |V|\ cos\ (2\pi\nu\frac{\vec{b}.\vec{\sigma}}{c} - \phi_V)$

Examining the assumptions

- Quasi-monochromatic radiation
  - Break the observing bandwidth into a large number of small spectral channels
  - τ_g can be compensated for only the direction of the phase centre ($\vec{s_0}$)
  - Imposes a limit on the size of useful FoV
- Ignore polarization
  - From a scalar, E_ν becomes a 2D vector and the Spatial coherence function a 2x2 matrix
- Sources at infinity
  - Mostly true
  - Possible to enhance the framework to deal with situations where the rays incident on the two antennas are not parallel

Examining the assumptions

- Empty celestial sphere
  - IGM, ISM, IPM
- Spatially incoherent radiation
  - Generally true
  - Execptions - masers, pulsars - both spatially unresolved
  - Coherence can be produced by scattering

Effect of bandwidth

- $dr = A_0 \, |V| \, cos(2\pi\nu\tau_g - \phi_V) \, d\nu$
- $r = A_0 \, |V| \, \int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} cos(2\pi\nu\tau_g - \phi_V) \, d\nu$
- $r = A_0 \, |V| \, \frac{sin \, \pi\Delta\nu\tau_g}{\pi\Delta\nu\tau_g} \, cos(2\pi\nu_0\tau_g - \phi_V)$

Frequency conversion/Heterodyning

- Frequency Conversion (mixing) - bringing the signal to an easier to handle (lower) frequency
  $\nu_{RF} = \nu_{LO} \pm \nu_{IF}$
  ν_{LO} = Local Oscillator (single frequency)
  ν_{IF} = Intermediate frequency (band)

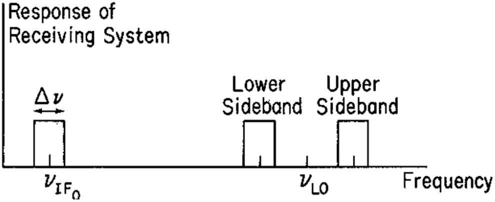
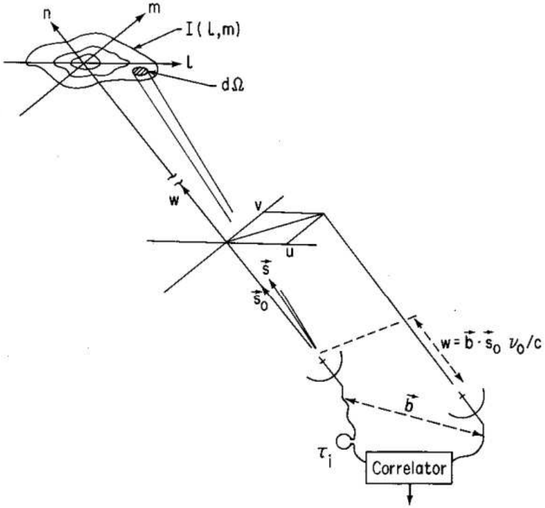


Figure 2–5. Relationship of RF (upper and lower sideband), IF, and LO frequencies.

Complex Correlator

- Complex Correlator
  - Correlator output - $r(\tau_g) =< V_1(t)\ V_2(t) >$
  - Need to compute both the cosine and sine components of the correlator output
  - $V_1(t) = v_1\ cos\ 2\pi\nu(t - \tau_g);\ V_2(t) = v_2\ cos\ 2\pi\nu t;$
    - $r(\tau_g) = v_1 v_2\ cos\ 2\pi\nu\tau_g$
  - $V_1(t) = v_1\ cos\ 2\pi\nu(t - \tau_g);\ V_2(t) = v_2\ cos\ 2\pi\nu t + \pi/2;$
    - $r(\tau_g) = v_1 v_2\ sin\ 2\pi\nu\tau_g$
- Delay tracking/Fringe stopping - automated compensation for τ_g

Coordinate System for Imaging



Coordinate System for Imaging

- $\vec{D_\lambda} \cdot \vec{s_0} = w$
- $\vec{D_\lambda} \cdot \vec{s} = (ul + vm + wn);\ n = \sqrt{1 - l^2 - m^2}$
- $d\Omega = \frac{dl\ dm}{\sqrt{1-l^2-m^2}}$
- $\vec{s} = \vec{s_0} + \vec{\sigma}$
  $\implies \vec{D_\lambda} \cdot \vec{\sigma} = \vec{D_\lambda} \cdot \vec{s} - \vec{D_\lambda} \cdot \vec{s_0}$
- $\vec{D_\lambda} \cdot \vec{\sigma} = ul + vm + w(\sqrt{1 - l^2 - m^2} - 1)$
- $\mathcal{V}(u, v, w) =$
  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_N(l, m)\ B(l, m)\ e^{-i2\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} \frac{dl\ dm}{\sqrt{1-l^2-m^2}}$

Small FoV approximation

- $w(\sqrt{1-l^2-m^2}-1) \sim -\frac{1}{2}(l^2+m^2)w << ul+vm$
- $\mathcal{V}(u,v,w) \sim \mathcal{V}(u,v,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_N(l,m)\ B(l,m)\ e^{-i2\pi[ul+vm]}\frac{dl\ dm}{\sqrt{1-l^2-m^2}}$

Impact of the w term

- Phase error - $\Delta\phi = \pi \; w \; (l^2 + m^2)$

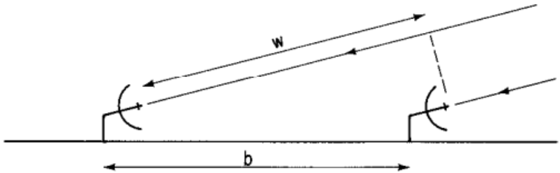


Figure 2-10. Comparison of the w-component and the antenna spacing when the direction of the source is close to that of the baseline. This condition can occur when the source is rising or setting.

- $\dfrac{1}{\theta_{HPBW}} \sim \dfrac{b_{max}}{\lambda} \sim w_{max}$; θ_{HPBW}- Synthesised Beam
- $\Delta\phi_{max} \sim \pi \left(\dfrac{\theta_F}{2}\right)^2 \dfrac{1}{\theta_{HPBW}}$; θ_F - size of the Map

Earth Rotation Synthesis Geometry

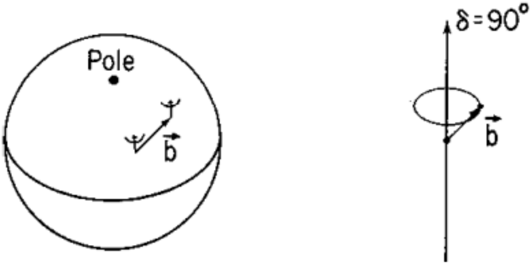


Figure 2-8. As the Earth rotates, the baseline vector **b**, which represents the spacing of the two antennas, traces out a circular locus in a plane normal to the direction of declination (δ) equal to $90°$. If the antennas are in an East–West line on the Earth, then the vector **b** is normal to the rotation axis.

Coordinate Frame

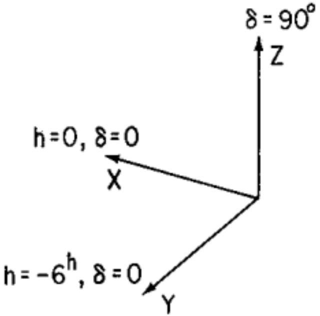


Figure 2-11. Coordinate system for specification of baseline parameters. X is the direction of the meridian at the celestial equator, Y is toward the East, and Z toward the North celestial pole.

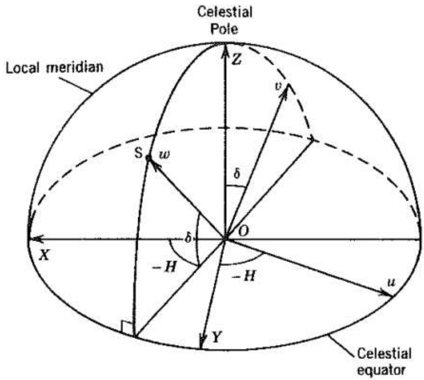


Figure 4.2 Relationships between the (X, Y, Z) and (u, v, w) coordinate systems. The (u, v, w) system is defined for observation in the direction of the point S, which has hour angle and declination H and δ. As shown, S is in the eastern half of the hemisphere and H is therefore negative. The direction cosines in the transformation matrix in Eq. (4.1) follow from the relationships in this diagram. The relationship in Eq. (4.2) can also be derived if we let S represent the direction of the baseline and put the baseline coordinates (h, d) for (H, δ).

Antenna Spacing Coordinates and *u, v, w*

($\delta = 90°$) for Z may be used as in Figure 2-11. Then if L_X, L_Y, and L_Z are the corresponding coordinate differences for two antennas, the baseline components (u, v, w) are given by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \sin H_0 & \cos H_0 & 0 \\ -\sin\delta_0 \cos H_0 & \sin\delta_0 \sin H_0 & \cos\delta_0 \\ \cos\delta_0 \cos H_0 & -\cos\delta_0 \sin H_0 & \sin\delta_0 \end{pmatrix} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}, \quad (2\text{-}30)$$

where H_0 and δ_0 are the hour-angle and declination of the phase reference position, and λ is the wavelength corresponding to the center frequency of the receiving system. The elements in the transformation matrix in Equation 2-30 are the direction cosines of the (u, v, w) axes relative to (X, Y, Z) axes: for further details see, e.g., Thompson, Moran and Swenson (1986). By eliminating

Comparatively recent developments

- Beyond the van Cittert Zernike theorem – The Fourier transform of the spatial coherence function of an incoherent source is equal to its complex visibility.
  - The sensitivity of practically modern instruments is high enough that under most circumstances ignoring the w term make the associated artifacts the dominant source of errors in the images
  - Mathematically well posed, but computationally more demanding, software implementations available (CASA, wsclean)
- Wide-FoV imaging and direction dependent calibration
  - Full polarization antenna beams
  - Propagation effects (ionosphere at low frequencies
  - Antenna to antenna differences

References

- Chap. 1 and 2, Synthesis Imaging in Radio Astronomy, ASPC Conf. Series Vol 6
- Chap. 2 and 4, Low Frequency Radio Astronomy