Sensitivity

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Signals and noise: two-element interferometer

Let us consider two antennas, i and j with equal gains G, but possibly different $T_{\rm sys}$

The voltages at the output terminals of the antennas are $v_i(t) = s_i(t) + n_i(t)$ and $v_j(t) = s_j(t) + n_j(t)$

The noise variances are $\sigma_i^2 = \langle n_i^2(t) \rangle = T^i_{\rm sys}$ and $\sigma_j^2 = \langle n_j^2(t) \rangle = T^j_{\rm sys}$

Signal power spectral density is $\langle s_i^2(t) \rangle = \langle s_j^2(t) \rangle = GS$

Consider an ordinary correlator: $r_{ij}(t) = v_i(t)\, v_j(t) = (s_i(t) + n_i(t))(s_j(t) + n_j(t))$ is the instantaneous output.

Mean output of correlator (ensemble average): $\langle r_{ij}(t) \rangle = \langle v_i(t)\, v_j(t) \rangle = \langle (s_i(t) + n_i(t))(s_j(t) + n_j(t)) \rangle$

$$= \langle s_i^2(t) \rangle = \langle s_j^2(t) \rangle = GS$$

We have assumed that the noise voltages are uncorrelated with each other, and the signals are uncorrelated with the noise (unbiased estimator of visibility).

But to estimate the noise in $r_{ij}(t)$ we need the root mean square of itself, I.e. $\langle r_{ij}(t) r_{ij}(t) \rangle$

$$\langle r_{ij}(t) r_{ij}(t) \rangle = \langle (v_i + n_i)(v_j + n_j)(v_i + n_i)(v_j + n_j) \rangle$$

Variance of correlator output

The quantity $\langle r_{ij}(t) r_{ij}(t) \rangle = \langle (v_i + n_i)(v_j + n_j)(v_i + n_i)(v_j + n_j) \rangle$ is difficult to compute.

For gaussian signals, fourth moment = $\sum$(product of second moments).

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\begin{aligned}
\langle r_{ij}(t) r_{ij}(t) \rangle &= 3(GS)^2 + (\sigma_i^2 + \sigma_j^2) GS + \sigma_i^2 \sigma_j^2 \\
&= 2(GS)^2 + (GS)^2 + \sigma_i^2 GS + \sigma_j^2 GS + \sigma_i^2 \sigma_j^2 \\
&= 2(GS)^2 + (GS + \sigma_i^2)(GS + \sigma_j^2) \\
&= 2(GS)^2 + (GS + T^i_{\rm sys})(GS + T^j_{\rm sys})
\end{aligned}
$$

Mean output of correlator (ensemble average): $\langle r_{ij}(t) \rangle = \langle v_i(t)\, v_j(t) \rangle = \langle (s_i(t) + n_i(t))(s_j(t) + n_j(t)) \rangle$

$$= \langle s_i^2(t) \rangle = \langle s_j^2(t) \rangle = GS$$

Variance of correlator output: $\mathrm{Var}(r_{ij}(t)) = \langle r_{ij}(t) r_{ij}(t) \rangle - \langle r_{ij}^2 \rangle = (GS)^2 + (GS + T^i_{\rm sys})(GS + T^j_{\rm sys})$

Time averaging vs ensemble averaging

$\langle\, . \,\rangle$ denotes ensemble-averaging; in real life we do time-averaging.

Time-averaged correlator output is: $\bar{r}_{ij}(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} r_{ij}(t')\ dt'$

So, our second moment becomes $\bar{\sigma}_{ij}^2 = \langle \bar{r}_{ij}\bar{r}_{ij}\rangle - \langle \bar{r}_{ij}\rangle^2$

For stationary signals,

$$\bar{\sigma}_x^2 = \frac{1}{T}\int_{-T/2}^{T/2}\left(1 - \frac{|\tau|}{T}\right) R_{xx}(\tau)\ d\tau$$

For quasi-sinusoidal signals of BW $\Delta\nu$, coherence time $\sim 1/\Delta\nu$

For long integration, $T \gg 1/\Delta\nu$; so $\tau/T \ll 1$

$$\bar{\sigma}_x^2 \simeq \frac{1}{T}\int_{-T/2}^{T/2} R_{xx}(\tau)\ d\tau \simeq \frac{1}{T}\int_{-\infty}^{\infty} R_{xx}(\tau)\ d\tau = \frac{1}{T}S_{xx}(0) = \frac{\sigma_x^2}{2T\Delta\nu}$$

Autocorrelation function:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\ x(\tau - t)\ dt$$

Fourier relation between autocorrelation function and power spectrum density: Parseval's theorem

$$R_{xx}(\tau) \Longleftrightarrow 2\Delta\nu\ S_{xx}(\nu)$$

Signal-to-noise ratio of a two-element interferometer

Noise variance: $\bar{\sigma}_x^2 \simeq \dfrac{\sigma_x^2}{2T\Delta\nu}$; $\sigma_{ij}^2(t) = (GS)^2 + (GS + T_{\text{sys}}^i)(GS + T_{\text{sys}}^j)$

Signal power: GS

$$S/N = \frac{\text{Signal power}}{\text{Noise RMS}} = \frac{GS}{\sqrt{\dfrac{(GS)^2 + (GS + T_{\text{sys}}^i)(GS + T_{\text{sys}}^j)}{2T\Delta\nu}}}$$

$$S/N = \frac{GS\sqrt{2T\Delta\nu}}{\sqrt{(GS)^2 + (GS + \mathrm{T}_{\text{sys}}^{\mathrm{i}})(\mathrm{GS} + \mathrm{T}_{\text{sys}}^{\mathrm{j}})}}$$

Two distinct cases

$$S/N = \frac{GS\sqrt{2T\Delta\nu}}{\sqrt{(GS)^2 + (GS + \mathrm{T}^{\mathrm{i}}_{\mathrm{sys}})(\mathrm{GS} + \mathrm{T}^{\mathrm{j}}_{\mathrm{sys}})}}$$

$GS \ll T_{\mathrm{sys}}$	$GS \gg T_{\mathrm{sys}}$
$S/N = \frac{\sqrt{2T\Delta\nu}GS}{T_{\mathrm{sys}}}$ Worse than a 2X antenna	$S/N = \frac{\sqrt{2T\Delta\nu}GS}{\sqrt{2(GS)^2}} = \sqrt{T\Delta\nu}$ Square root (N)

Signal-to-noise ratio of an N-element interferometer

N-element interferometer = ${}^{N}C_2$ two-element interferometers, where ${}^{N}C_2 = \dfrac{N(N-1)}{2}$

- Any two two-element interferometers with one common antenna : noise is uncorrelated
- Any two two-element interferometers with no common antenna : (also) noise is uncorrelated

Noise from each two-element interferometer can be added in quadrature.

$$S/N = \frac{\sqrt{N(N-1)T\Delta\nu}\,GS}{T_{\rm sys}}$$

For N dishes, it would have been $N/\sqrt{N(N-1)}$ better.

GMRT: N=30 => only 1.02 better

Noise in the image plane

$$I(l,m) = \frac{1}{M}\sum_p w_p V_p e^{-i2\pi(lu_p+mv_p)}$$

$$\langle I(l,m)\ I(l',m')\rangle = \frac{1}{M^2}\sum_p \sum_q w_p w_q \langle V_p V_q^* \rangle e^{-i2\pi(lu_p+mv_p)} e^{i2\pi(l'u_q+m'v_q)}$$

$$\langle I(l,m)\ I(l',m')\rangle = \frac{1}{M^2}\sum_m \sigma_p^2 e^{-i2\pi((l-l')u_p+(m-m')v_p)} = \frac{\sigma_p^2}{M}$$

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Questions